Deformations of Algebras

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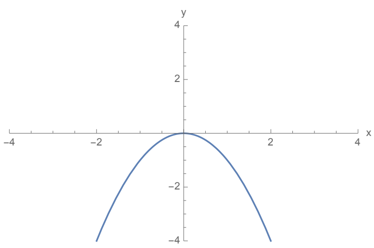
This essay is a non-technical summary of my masters degree project titled *Deformations of Algebras*. This project was an introduction to algebraic deformation theory and cohomological aspects of deformations of algebras and the proof Coffee's Theorem.

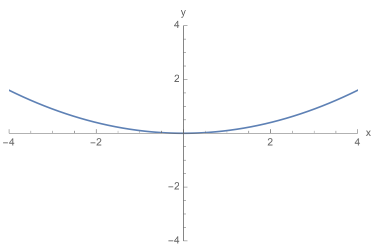
Deformations of algebras gained fame in pure mathematics when they were used in the proof of Fermat's Last Theorem. Prior to this application of deformations of algebras, the theory was developed by algebraist Murray Gerstenhaber, who wrote a 5-paper long treatise on the subject in the 1960's-70's. These papers are considered the beginning of algebraic deformation theory. Outside of mathematics there has been a growing interest in algebraic deformation theory from quantum physicists. Such physicists study a phenomenon called "deformation-quantization"[7], which has its roots in algebraic deformation theory (however it is unclear which came first; algebraic deformation theory or deformation-quantization). Whether you are aiming to try and understand the proof of Fermat's Last Theorem, or if you are a keen quantum physicist, you will hopefully enjoy the following introduction to deformation theory.

In order to define a deformation of an algebra we first define an algebra. There are many ways to define an algebra, and they are not all equivalent, however for our purposes an **algebra** is as a set of elements, A, which you can **add** and **multiply**, together with a set of scalars, k, which you can **scale** elements with.

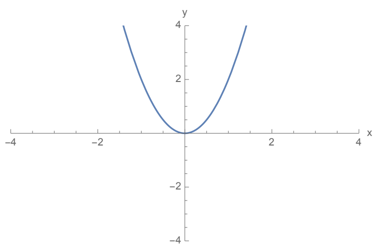
A **deformation** of an algebra, A, is a set of algebras, $\{A(t) \mid t \in k\}$, such that $A(0)$ is the same as (or rather, isomorphic to) the original algebra A. It is not immediately clear why this definition has the name "deformation", however when deforming so-called "polynomial algebras" one can interpret this algebraic definition geometrically and therefore more intuitively. Every polynomial algebra corresponds to a geometric shape, known as its spectrum, and as you deform a polynomial algebra, the corresponding spectrum also deforms geometrically, thus inspiring the name "deformation". You can see an example of the deformation of a spectrum in the figure below, where the blue curve is the spectrum of a given algebra. As we deform this algebra, the curve deforms too but in a geometric sense, ie. the curve is bending.



Spectrum for $t = -1$

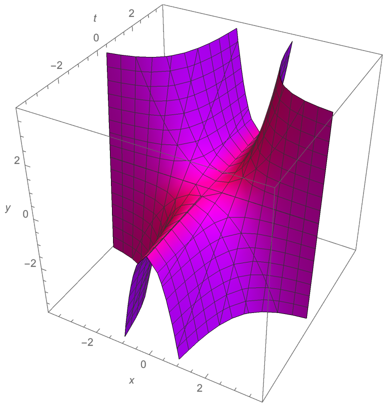


Spectrum for $t = \frac{1}{10}$.



Spectrum for $t = 2$.

Another geometric way to view a deformation of a polynomial algebra is to look at the surface traced out by all of the spectra at once, as shown on the next page.



Spectrum of the entire deformation.

Deformations of Algebras does not follow the geometric study of deformations, however being able to view a mathematical object often helps understanding the underlying theory.

Next we introduce **Hochschild cohomology**, a tool for investigating the deformations of a given algebra. Hochschild cohomology is an instance of a broader theory known as "homological algebra", where one studies methods of associating algebraic data to mathematical objects. Roughly speaking, when studying cohomology, we take some mathematical object X (such as a shape, a space, an algebra, a set of functions etc), and associate to it a set of algebraic objects $\{C^n(X) \mid n = 1, 2, \ldots\}$. With these objects we may calculate **cohomologies**, denoted by $H^n(X)$ for $n = 1, 2, \ldots$. For particular sets of scalars, k, we can also calculate the "dimension" of $H^n(A)$ for any whole number n. The goal of this process is to use this algebraic data (ie. $H^n(X), C^n(X)$) to understand the original object X. There are various types of cohomology, and algebraists spend whole research-careers trying to understand them, but for the purposes of this essay we only need to know that Hochschild cohomology takes algebras as its input, and outputs practical data about them. An example of such data is that for an algebra A, the second Hochschild cohomology, $H^2(A)$, is the set of deformations of A.

The reason we use cohomology in mathematics is that it is often easier to compute cohomologies of X than it is to reason about X directly. In fact, one of the first results presented in *Deformations of Algebras* is a good example of this, where it is shown that we can check whether an algebra A has any deformations at all purely by calculating $H^2(A)$.

The main result in *Deformations of Algebras* was originally proved by one of Gerstenhaber's students, Jane Purcel Coffee, in the 1970's, and it is a cohomological criterion to for a filtered algebra to be graded. We should first make sense of what "filtered" and "graded" mean before continuing. An algebra A is **filtered** if we can find subsets $F_n \subseteq A$ for $n = 1, 2, \ldots$ such that each subset is itself an algebra, and if we can order them in the following way:

$$A \supseteq F_1 \supseteq F_2 \supseteq \cdots .$$

This is called a "chain of subalgebras", and the if A is filtered, we require that elements in A move down the chain of subalgebras when we multiply them[1].

An algebra is **graded** if it is filtered and if every element in the algebra can be written uniquely as a sum of simpler elements[2]. In practice, you are spared a lot of work when dealing with graded algebras, so it is in your best interest to work out whether or not your algebras are graded. However, showing that an algebra is graded can be difficult, hence having computational criteria to tell us whether or not an algebra is graded is warranted.

[1]For algebraists: an algebra A is filtered if there exists a chain $A = F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots$ such that for any $m, n \in \mathbb{N}$ we have $\forall a \in F_n, \forall b \in F_m, \Rightarrow ab \in F_{n+m}$.

[2]For algebraists: an algebra A is graded if it has a filtration $(F_i)_{i \in \mathbb{N}}$ *and* we have $A \cong \prod_{i \in \mathbb{N}} A_i$, where $A_i = F_i / F_{i+1}$ for all $i \in \mathbb{N}$.

We conclude this essay by seeing what Coffee's Theorem does. Coffee's Theorem[3] lets us check whether a filtered algebra A is graded or not, by computing two particular cohomologies of A, and if the dimensions of these two cohomologies are equal, then A is graded. The only drawback of Coffee's Theorem is that it only holds for algebras with a particular set of scalars k. In spite of this, Coffee's Theorem is a useful result in both theory, and in practice. *Deformations of Algebras* culminates in a proof of Coffee's Theorem, using all of the above deformation theory and Hochschild cohomology, which may surprise you when noting that deformations of algebras are not mentioned at all in this paragraph (nor in the full statement in footnote 3). The proof of Coffee's Theorem is in fact entirely an exercise in deformation theory even though the statement does not suggest so.

Hopefully this has been an enjoyable summary of *Deformations of Algebras*. If the reader would like to further their knowledge of algebraic deformation theory, a good place to start Fox's brief overview of algebraic deformation theory[6], and then follow with Gerstenhaber's original papers [1, 2, 3, 4]. Deformations of algebras is a simple, yet fruitful theory and has proven its value both inside and outside of mathematics, and by the end of this essay you too have seen why this is the case.

References

[1] M. Gerstenhaber, [1964], *On the Deformations of Rings and Algebras*, Annals of Mathematics, Second Series, Vol.79.

[2] M. Gerstenhaber, [1966], *On the Deformations of Rings and Algebras II*, Annals of Mathematics, Second Series, Vol.84.

[3] M. Gerstenhaber, [1968], *On the Deformations of Rings and Algebras III*, Annals of Mathematics, Second Series, Vol.88.

[4] M. Gerstenhaber, [1974], *On the Deformations of Rings and Algebras IV*, Annals of Mathematics, Second Series, Vol.99.

[5] J.P. Coffee, [1972], *Filtered and associated graded rings*. Bull. Amer. Math. Soc. 78, no. 4, 584–587. https://projecteuclid.org/euclid.bams/1183533903

[6] T.F. Fox, [1993], *An Introduction to Algebraic Deformation Theory*, Journal of Pure and Applied Algebra, Vol.84.

[7] A.Giaquinto, [2010], *Topics in Algebraic Deformation Theory*, arXiv:1011.1299.

[3] **Coffee's Theorem**: Let A be a separeted complete filtered algebra over a field k of characteristic 0. If $\dim_k H^2(A) = \dim_k H^2(\mathrm{gr}A)$ and is finite, then $A \cong \mathrm{gr}A$, where $\mathrm{gr}A$ is the associated graded algebra of A.