Deformations of Algebras

Contents

Abstract

This essay will introduce the theory of deformations of algebras and the appropriate related homological algebra. This essay is inspired by Murray Gerstenhaber's defining papers on the subject [2, 3, 4, 5] as well as Fox's survey on deformations of algebras [7] and is pitched at the level of a starting graduate student. To the end of the essay we go on to study the deformation theory of filtered and graded associative algebras and will develop a purely cohomological condition through the deformation theory developed by Coffee [6] to guarantee that a filtered algebra is graded.

0 Preliminaries

Throughout this essay we will be using the following definitions and notation:

- Given a commutative ring R with unit 1 and (left) R-modules M, N, P, we say that a map $\phi : M \to N$ is R-**linear** if ϕ is a homomorphism of abelian groups and if $\forall m \in M$ and $\forall r \in R$ we have $\phi(rm) = r\phi(m)$. We say that a map $\theta : M \times N \to P$ is R-**bilinear** if θ is a group homomorphism $M \times N \to P$ and if for any $m \in M$ and any $n \in N$ the maps $\theta(m, -) : N \to P$ and $\theta(-, n) : M \to P$ are R-linear.

- We say that an **algebra** A over a (commutative) ring R is an R-module with a multiplication $A \times A \to A$ which distributes over the addition from the R-module structure, and is R-bilinear. We say that A is an **associative algebra** if the multiplication on A is associative, i.e. if $\forall a, b, c \in A$ we have $(ab)c = a(bc)$. I will use the symbol 1_A for the identity morphism $1_A : A \to A$ defined by $a \mapsto a$ for all $a \in A$.

- Given a ring R, the **formal power series ring** $R[[t]]$ is an R-algebra consisting of elements $\sum_{i \geq 0} t^i a_i$, with addition

$$\sum_{i \geq 0} t^i a_i + \sum_{i \geq 0} t^i b_i := \sum_{i \geq 0} t^i (a_i + b_i),$$

and multiplication

$$\left(\sum_{i \geq 0} t^i a_i \right) \left(\sum_{j \geq 0} t^j b_j \right) := \sum_{m \geq 0} \sum_{\substack{i+j=m \\ i,j \geq 0}} t^m (a_i b_j),$$

and R-action

$$r \left(\sum_{i \geq 0} t^i a_i \right) := \left(\sum_{i \geq 0} t^i (r a_i) \right),$$

for all $\sum_{i \geq 0} t^i a_i, \sum_{i \geq 0} t^i b_i \in R[[t]]$ and all $r \in R$. Note that this is all *formal*, so we don't worry about these sums converging. Set-theoretically we may consider $R[[t]]$ as the set of sequences $\{(a_i)_{i=0}^{\infty} \mid a_i \in R \, \forall i \geq 0\}$. In fact, if we forget the multiplication on $R[[t]]$ we have a R-module isomorphism between $R[[t]]$ and $\prod_{i \geq 0} R$ with entrywise addition.

1 Introduction

Before we get started properly, here is a short explanation of the structure of this essay. There are four parts to this essay. In part 0 contains preliminary definitions and notation.

In part 1 of this essay we will introduce the central objects of study in deformation theory, **deformations of algebras**, and the related homological algebra for deformations of associative algebras, namely **Hochschild Cohomology**.

In part 2 we also have two sections, the first of which introduces filtered and graded associative algebras and their deformation theory over a field of characteristic zero. The second section contains the main result of this essay regarding deformations of filtered and graded algebras over a field of characteristic zero which follows Coffee's short paper on the subject [6].

We conclude this essay with a brief summary in part 3.

Deformations

In this section we introduce deformation theory in a hands-on context through plenty of examples and explicit calculations to try and give a concrete idea of what we mean when we refer to things like the deformation theory of an algebra.

Before defining deformations rigorously we will try and motivate why we should care about deformations in the first place. One reason to study deformations of algebras is to get a background understanding of *deformation-quantization* [8]. Physicists tend to care a great deal about all things quantum, and it turns out that the theory of deformations of algebras leads naturally to the study of deformation-quantization. A more mathematical reason to study deformation theory is to answer the question "*how can I make a commutative algebra into a noncommutative one*?". Deformations give us examples of this. For example, for any field k, consider the algebra $A := k[x, y]$. We have $xy = yx$ in this polynomial algebra, but what now if we impose a new relation on A by letting $xy := 2yx$ for example. Let us for now denote the algebra $k[x, y]$ with the relation $xy = 2yx$ by the symbol A_1. Clearly$A_1 \cong A/(xy - 2yx)$.

Once we make the proper definition of deformation we will see that A_1 is an instance of an entire family of algebras A_t parametrised by $t \in k$ where

$$A_t := A[t]/(xy - (t + 1)yx).$$

In [7] Fox introduces deformations of an algebra A as a one-parameter family of algebras A_t where t varies through k such that $A_0 \cong A$. For us this will serve as an intuition rather than definition as we will require a stronger definition to study deformations of filtered and graded algebras. However it is still worthwhile to have in mind the idea that a deformation of an algebra is a family of algebras that are parametrised by the groundfield (or groundring). I find this particularly useful to have in mind once we start using the more useful definition, where this idea is less obvious.

From this point onwards, unless stated otherwise, we will let k be a field and we will let A be an algebra over k. We may say algebra rather ambiguously for now, but we will specify to associative algebras fairly soon. The reason we say "an algebra" so vaguely is to let the reader know that there is a deformation theory for algebras general types of algebras forming so called "categories of interest" [3], which involves an abstraction into a category-theoretic setting. Such an increase in generality comes with some loss of detail, so as stated previously we will be considering the deformation theory of associative algebras for the majority of this essay to get a firm idea of deformation theory. We are now ready to learn the formal definition of a deformation of an algebra.

Definition. *Given an algebra A over some field k (or for a ring viewed as an algebra over itself), a* **one-parameter family of deformations** *or simply a* **deformation** *of A is a family of multiplications f_t on $A[[t]]$ for some parameter t varying over k. By "multiplication on $A[[t]]$" we mean a $k[[t]]$-bilinear map $f_t : A[[t]] \times A[[t]] \to A[[t]]$ in the form of a formal power series:*

$$f_t = \pi + tF_1 + t^2F_2 + \cdots$$

where π (also written as F_0) denotes the original multiplication in A, and for each $i \geq 1$, F_i is a $k[[t]]$-bilinear map $A[[t]] \times A[[t]] \to A[[t]]$.

We may also refer to the one-parameter family of deformations by the symbols A_t when f_t is understood, or by A_f. The reason behind this alternate notation is to emphasise that a deformation f_t gives rise to a new algebra related to the given algebra A.

We also require that a deformation satisfies the same relations as the original multiplication on an algebra, so if A is associative/Lie/Jordan etc. then so is A_t.

Remark. With this definition of one-parameter family of deformations we actually have that A is embedded in the family A_t by evaluating $t = 0$. Clearly, when $t = 0$ we have that $f_0 = \pi$, and $A[[0]] \cong A$ and so $A_0 \cong A$, coinciding with Fox's notion of deformation.

One may also ask why we deform the multiplication of an algebra and not some other structure. There is no direct answer to this in the literature, however one immediate problem would be that deforming the only other structure of an algebra, the module structure, may result in a loss in commutativity in the addition of the algebra and so we would no longer even *have* an algebra left after deformation. Hence it makes sense for us to deform the multiplication of an algebra rather than any other of its structures.

Example 1. *To see what is meant by requiring a deformation to satisfy the same relations as the original algebra we look at deformations of associative and of Lie algebras here. For example, if A is an associative algebra we require $\forall a, b, c \in A$*

$$f_t(f_t(a, b), c) = f_t(a, f_t(b, c)). \tag{1}$$

If A is a Lie algebra with deformation f_t we have again $\forall a, b, c \in A$

$$f_t(a, b) = -f_t(b, a) \text{ and } f_t(a, f_t(b, c)) + f_t(c, f_t(a, b)) + f_t(b, f_t(c, a)) = 0 \tag{2}$$

We will be focusing mainly on associative algebras, however much of the theory here holds for associative, Lie, Jordan, Poisson and Bialgebras [2], or as mentioned earlier, any category of algebras that form a "category of interest" [3].

Note. It is no mistake that we have written $\forall a, b, c \in A$ rather than "$\in A[[t]]$". This is because we have defined our one-parameter families of deformations in terms of sums of $k[[t]]$-bilinear maps $F_i : A[[t]] \times A[[t]] \to A[[t]]$ (rather than k-bilinear maps). That is, the terms of our deformations are defined on $A \times A$ and extended $k[[t]]$-linearly. Note also that once we have extended to $k[[t]]$-bilinear maps, we may canonically extend further to $k((t))$-bilinear maps, where $k((t))$ the field of fractions of $k[[t]]$. We make the remark now but will not need it until we define Hochschild cohomology where $k((t))$-bilinearity will allow us to use vector space dimensions of "Hochschild cohomology groups".

At this stage it may not be entirely intuitive why deformations of algebras have the name "deformation". In my mind the word deformation is reserved for geometric or analytical objects, which does not immediately mesh with the definition we have above. The following examples of deformations of polynomial algebras will establish a link between this geometric intuition of deformations, and the definition of deformations of algebras.

Example 2.

- *Consider the algebra*[1] $A := \mathbb{R}[x]/(x^2)$. *Here we have a rather boring spectrum with* $\mathrm{Spec}(A)$ *being a single point* (x) *(inside the the line* $\mathrm{Spec}(\mathbb{R}[x])$*). Then let* $f_t = \pi + t^2 F$*, where* $F(x, x) = t^2$ *and zero otherwise. Thus* $A_t \cong k[x, t]/(x^2 - t^2)$*, and so we have that* $\mathrm{Spec}(A_t) = \{(x-t), (x+t)\}$ *i.e. two points varying with* $t \in \mathbb{R}$*, and when* $t = 0$ *then* $\mathrm{Spec}(A_t) = \mathrm{Spec}(A_0)$ *becomes a single point again.*

- *Consider the algebra* $A = \mathbb{R}[x, y]/(x^2)$. $\mathrm{Spec}(A)$ *is a line which we may think of as an axis of* $\mathbb{R}^2$*. For our first one-parameter family of deformations we have the multiplication* f_t *on* $A[[t]]$ *defined on the generators of* $A[[t]] \times A[[t]]$ *to be:*

$$\begin{array}{lll} f_t(1,1) := 1, & f_t(1,y) = f_t(y,1) := y, & f_t(1,x) = f_t(x,1) := x \\ f_t(x,x) := ty, & f_t(x,y) = f_t(y,x) = xy = yx & \\ f_t(y,y) := y^2 & & \end{array}$$

Upon extending f_t *linearly we have that* f_t *is a multiplication on* $A[[t]]$*. Writing* f_t *in terms of a sum of* $\mathbb{R}$*-bilinear maps we have* $f_t = \pi + tF$*, where* $F(x, x) = y$ *and is zero otherwise. If we wish to do away with the symbol* f_t *altogether we may also observe that*

$$A_t \cong \mathbb{R}[x, y, t]/(x^2 - ty).$$

To see what this deformation does to the line we look at instances of A_t *for* $t = -1, \frac{1}{10}, 2$*. We see plots of the spectra of* A_t *for these values in figures 1,2 and 3.*

[1] We work over $\mathbb{R}$ here so that we may draw pictures. Of course we could work over an arbitrary field here.

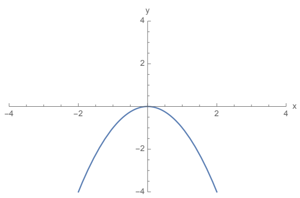

Figure 1: $\mathrm{Spec}A_{-1}$

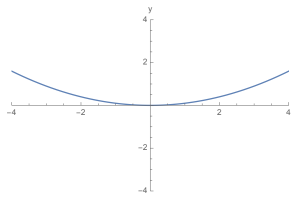

Figure 2: $\mathrm{Spec}A_{\frac{1}{10}}$

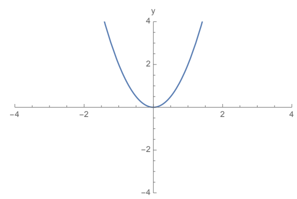

Figure 3: $\mathrm{Spec}A_2$

We even have that the spectrum of A_t when considering t as a variable gives us the surface in figure 4:

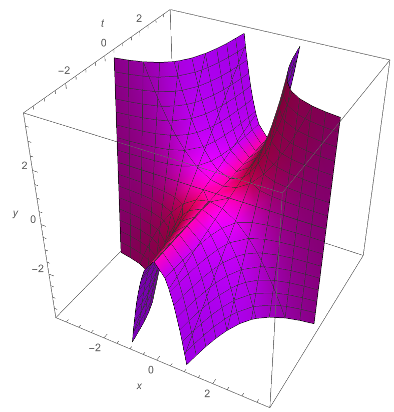

Figure 4: Spectrum of the family A_t varying through t.

- *Lastly we consider $B = \mathbb{R}[x,y]/(y^2 - x^3)$. We see that $\mathrm{Spec}(B)$ is a cuspidal cubic curve in the plane. Now instead of explicitly defining a multiplication on $B[[t]]$ we may simply define a deformation B_t of B by letting*

$$B_t := \mathbb{R}[x,y,t]/(y^2 - x^3 + t^2 x).$$

We can see the spectra of B_t for $t = 0,1,2$ in figures 5,6 and 7 respectively.

As in the previous example we may look at the surface given by the spectrum of B_t in figure 8.

Hopefully these examples have clarified what deformations of algebras can look like, and maybe even shed some intuition onto the definition of deformation. We proceed into the abstract by showing that if we fix an algebra A, the deformations of A form a category with deformations of A as objects and morphisms yet to be defined.

The "a priori" definition of morphism of deformations presented next is not a formal definition by any means, but it is supposed to look like a morphism of alge-

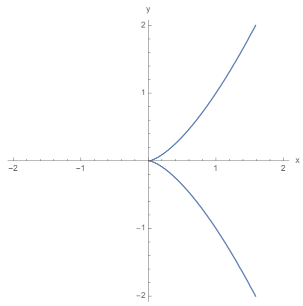

Figure 5: $\mathrm{Spec} B_0$

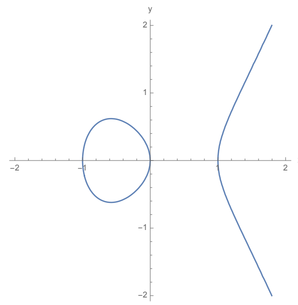

Figure 6: $\mathrm{Spec} B_1$

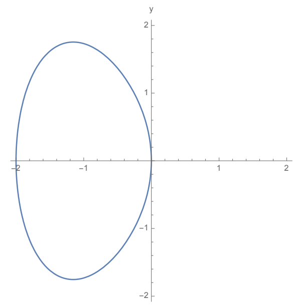

Figure 7: $\mathrm{Spec} B_2$

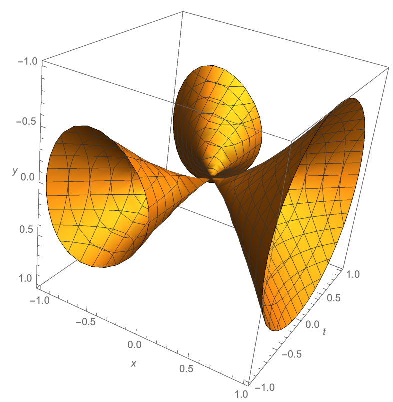

Figure 8: Spectrum of the family B_t varying through t.

bras. The reasoning behind this approach for defining morphisms of deformations is that it will motivate the more commonly used definition of *algebraic automorphisms*.

Definition ("A priori" defintion of morphisms of deformations)**.** *Given an algebra A over k and deformations g_t and f_t of A, a morphism of deformations $f_t \to g_t$ is a $k[[t]]$-linear map $\Psi : A_f \to A_g$ which satisfies:*

$$\Psi(f_t(a,b)) = g_t(\Psi(a), \Psi(b))$$

for all $a, b \in A$.

Remark. This definition should look just like a morphism of algebras as you have seen before, except that almost all spaces in the definition involve formal power series. Because of the prevalence of power series we have a particular way to express a morphism of deformations which we investigate next. Consider $\Psi : A_f \to A_g$ as in the definition. Since we require Ψ to be $k[[t]]$-linear we only need to define Ψ on elements in A (like we did in (1),(2)). So we are left to make sense of the expression $\Psi(a)$ for any $a \in A$. We have that $\Psi(a) \in A_g$, so we know that $\Psi(a)$ is a formal power series in t over A. Hence $\Psi(a) = b_0 + tb_1 + t^2 b_2 + \cdots$ for some

$b_i \in A$, $i \geq 0$. It should now feel intuitive to write $\Psi_t = \psi_0 + t\psi_1 + t^2\psi_2 + \cdots$ where each ψ_i is $k[[t]]$-linear. In terms of the previous expression of $\Psi(a)$ we now have $\psi_i(a) = b_i$ for each $i \geq 0$.

We now go further and note that for our one-parameter families of deformations f_t, g_t we have that F_0 and G_0 are just the original multiplication on A. This should be reflected in our definitions of morphisms too. By having $\Psi_0 : A_0 \to A_0$ being the identity morphism 1_A. Since $\Psi_0 = \psi_0$ it makes sense to require $\psi_0 = 1_A$. Thus any morphism of deformations Ψ_t is of the form

$$\Psi_t = 1_A + t\psi_1 + t^2\psi_2 + \cdots .$$

We will now see that any such map Ψ_t is in fact a bijection, and will lead to the definition of algebraic automorphism.

To show that Ψ_t has an inverse, we define a hypothetical inverse morphism $\Phi_t : A_g \to A_f$ of Ψ_t. We will then work backwards to show that one can define Φ_t purely in terms of the known components ψ_i.

We have $\Phi_t = 1_A + \sum_{j\geq 1} t^j \phi_j$. Thus we want to see that $\Psi_t\Phi_t = 1_A$ and $\Phi_t\Psi_t = 1_A$. Expanding the left hand sides in terms of formal power series gives us the following:

$$\begin{aligned}\Psi_t\Phi_t &= \left(1_A + \sum_{i\geq 1} t^i\psi_i\right)\left(1_A + \sum_{j\geq 1} t^j\psi_j\right)\\ &= 1_A + \sum_{n\geq 1}\sum_{i+j=n} t^n \psi_i\phi_j.\end{aligned}$$

Thus for $\Psi_t\Phi_t = 1_A$ we require that $\Psi_t\Phi_t(a) = a$ for all $a \in A$. Expanding this equation in terms of formal power series gives us:

$$a + \sum_{n\geq 1}\sum_{i+j=n} t^n \psi_i(\phi_j(a)) = a,$$

and so looking at the coefficients of t^n we have for each $n \geq 1$ that

$$\sum_{i+j=n} \psi_i(\phi_j(a)) = 0,$$

which allows us to define the ϕ_i inductively. For $n = 1$ we have

$$\begin{array}{lcl} & \psi_0(\phi_1(a)) + \psi_1(\phi_0(a)) & = 0 \\ \iff & \phi_1(a) & = -\psi_1(a). \end{array}$$

and so we let $\phi_1 := -\psi_1$. For $n = 2$ we have

$$\begin{array}{lcl} & \psi_0(\phi_2(a)) + \psi_1(\phi_1(a)) + \psi_2(\phi_0(a)) & = 0 \\ \iff & \phi_2(a) & = \phi_1^2(a) - \psi_2(a), \end{array}$$

and so we let $\phi_2 := \phi_1^2 - \psi_2$.

Continuing in this way we have for arbitrary $n > 0$ that

$$\phi_n := - \sum_{\substack{i+j=n \\ j \neq n}} \psi_i \phi_j.$$

This shows that we can *always* construct an inverse of a given morphism of deformations; making every "morphism" an isomorphism. Since the underlying algebra of both A_f and A_g is the same it is common to call these maps *automorphisms*[2] as we do now in the formal definition of equivalence of deformations.

Definition. *Two deformations f_t, g_t of an algebra A are said to be* **equivalent** *if there exists a* **formal linear automorphism** $\Psi_t : A[[t]] \to A[[t]]$ *of the form*

$$\Psi_t = 1 + t\psi_1 + t^2\psi_2 + \cdots$$

where $\psi_i : A_t \to A_t$ is $k[[t]]$-linear morphism for each $i \geq 1$ such that for any $a, b \in A$ we have

$$f_t(a, b) = \Psi_t^{-1}(g_t(\Psi_t(a), \Psi_t(b)))$$

which is commonly abbreviated to

$$f_t = g_t \circ \Psi_t.$$

We commonly abuse the notation and write $\Psi_t : A_f \to A_g$.

A deformation is said to be **trivial** *if it is equivalent to the original multiplication.*

Definition. *An algebra is said to be* **rigid** *if all its deformations are trivial.*

By now we have seen plenty of examples of deformations and have a notion of when two deformations are the same or when a deformation of an algebra is not much different from the original algebra. The following section introduces Hochschild cohomology which will turn out to be an incredibly useful tool for us to investigate deformations of associative algebras.

Hochschild Cohomology

I expect the reader of this essay to have as good a grasp of homological algebra as I did as a first year graduate - that is to say no idea at all. Homological algebra deserves plenty of attention in its own right and is the topic of plenty of essays, books and indeed entire courses or entire careers in mathematics. We will first see a brief hands-on summary of what we mean by homological algebra before quickly specialising to the necessary branch of cohomology for deformations of associative algebras, namely *Hochschild cohomology*.

Homological algebra is a method used to associate algebraic data to mathematical objects. This may be a vague statement, but the techniques and calculations involved are not so difficult to understand. As the names suggest, homology and

[2] I prefer to think of this as "automorphisms of deformations" since we already have a notion of automorphisms of algebras which are distinct to the automorphisms like $\Psi_t : A_f \to A_g$.

cohomology are dual (in the categorical sense) and so we will proceed by defining cohomology and then leave it to the reader to understand the dual theory by "reversing all the arrows" later (and removing the prefix "co-" from all the objects and morphisms we now define) if they so wish.

To define cohomology we in general take some mathematical object (for example a group, a topological space, a sheaf of rings over a topological space etc.) and somehow associate to it a **cocomplex**, which is a sequence of algebraic objects called **cochains** and morphisms called **coboundary operators**. Cochains may form abelian groups, algebras, R-modules etcetera, and the sequence they form is denoted $(C^n)_{n\in\mathbb{Z}}$. We refer to C^n as the group/algebra/module of n-**cochains**. Coboundary operators are denoted for each $n \in \mathbb{Z}$ by $\delta_n : C^n \to C^{n+1}$ and are homomorphisms of the corresponding structures (so if $(C^n)_{n\in\mathbb{Z}}$ are groups/algebras/R-modules, δ_n are group/algebra/R-linear homomorphisms). We then require that coboundary operators satisfy $\delta_n\delta_{n-1} = 0$. This can be expressed as $\operatorname{im}\delta_{n-1} \subseteq \ker\ \delta_n$ for each $n \in \mathbb{Z}$. We have the following diagram to remember it all by:

$$\cdots \xrightarrow{\delta_{n-2}} C^{n-1} \xrightarrow{\delta_{n-1}} C^n \quad \xrightarrow{\delta_n} C^{n+1} \xrightarrow{\delta_{n+1}} \cdots .$$

- The image of δ_{n-1} is called the set of n-**coboundaries**, and is denoted B^n.
- The kernel of δ_n is called the set of n-**cocycles**, and is denoted Z^n.
- The n-**th cohomology** is defined to be the quotient $H^n := Z^n/B^n$.

Elements of H^n are called **cohomology classes**. The symbol $H^\bullet$ denotes the cocomplex $(C^n, \delta_n)_{n\in\mathbb{Z}}$.

Remark. We will often abuse notation when referring to cohomology classes, for example if $F \in Z^n$ we may refer to the "cohomology class $F \in H^n$" by which we mean the image of F under the canonical projection $Z^n \twoheadrightarrow H^n$ (given by $F \mapsto F + B^n$).

The most complicated part of calculating cohomologies is the initial part where we somehow associate algebraic data to a mathematical object. The way we choose to associate this data to a mathematical object is in general quite laborious. Thankfully for us, forming the Hochschild cocomplex turns out to be fairly simple.

From this point on, we require all algebras to be associative.

Definition (Hochschild Cohomology)**.** *Fix an algebra A over some field k. We define the* **Hochschild cocomplex** *for each $n \geq 0$ as:*

- *The set of* **Hochschild n-cochains** *is $C^n(A, A) := \operatorname{Hom}_k(A^n, A)$, where we take $A^0 := k$, and $A^n := A \times A\ \times \cdots \times A$ n times for $n > 0$.*
- *The* **Hochschild coboundary operator** *of this complex for each $n \geq 0$ as $\delta_n : C^n(A, A) \to C^{n+1}(A, A)$ for any $F \in C^n$ as follows:*

$$(\delta_n F)(a_1, a_2, \cdots, a_n) := a_1 F(a_2, \cdots, a_n) + \sum_{i=1}^{n-1} (-1)^i F(a_1, \cdots, a_i a_{i+1}, a_{i+2}, \cdots a_n) + (-1)^n F(a_1, \cdots, a_{n-1}) a_n.$$

(Feel free to verify that $\delta_{n+1}\delta_n = 0$).

We will often write δ in place of δ_n for brevity. Note that δ_n is a homomorphism of k-modules for each $n \geq 0$.

The notation we use for the Hochschild n-cocycles, n-coboundaries and n-th cohomology are $B^n(A, A), Z^n(A, A)$ and $\mathrm{Hoch}^n(A, A) = Z^n(A, A)/B^n(A, A)$ respectively.

We often write H^n or $H^n(A, A)$ for $\mathrm{Hoch}^n(A, A)$ for brevity (some people also write $HH^n(A, A)$ [8]). It is conventional to refer to $H^n(A, A)$ as the "n-th Hochschild cohomology group", even though $H^n(A, A)$ is a k-module.

Note that we keep writing $H^n(A, A)$ rather than $H^n(A)$. This is the conventional notation for Hochschild cohomology, and for us this may be superfluous, however in general one may wish to consider a cocomplex with n-cochains of the form $\mathrm{Hom}_k(A^n, P)$ for some algebra P, which would be denoted $C^n(A, P)$ (and similarly we would have $Z^n(A, P), B^n(A, P)$ and $H^n(A, P)$).

Remark. We will now see that Hochschild Cohomology Cocomplexes have plenty of structure. In fact $H^\bullet(A, A)$ has a structure reminiscent of a Lie algebra.

We first define the "circle product" for any $n, m \geq 0$ mapping

$$\circ : C^n \times C^m \to C^{n+m-1}$$

defined by the following rather intricate composition of cochains: for any $F \in C^n$ and $G \in C^m$ we define for any $(a_1, \ldots, a_{n+m-1}) \in A^{n+m-1}$,

$$(F \circ G)(a_1, \ldots, a_{n+m-1}) := \sum_{i=1}^{n} (-1)^{(m-1)(i-1)} F(a_1, \ldots, a_{i-1}, G(a_i, \ldots, a_{i+m-1}), a_{i+m}, \ldots, a_{n+m-1}) \tag{3}$$

We then define a so called "graded pre-Lie bracket" for $n, m \geq 0$, $[\,,\,] : C^n \times C^m \to C^{n+m-1}$ for any $F \in C^n$ and any $G \in C^m$ as

$$[F, G] := F \circ G - (-1)^{(n-1)(m-1)} G \circ F.$$

Note that the circle (and therefore the bracket) both preserve coboundaries, i.e. the circle product of two coboundaries is a coboundary, hence confirming that the circle (and the bracket) are well-defined operations on the Hochschild cocomplex $H^\bullet(A, A)$.

We introduce this structure because it provides useful notation for later theory, namely the fact that for any n-cochain $F \in C^n(A, A)$ we have $\delta F = (-1)^{n-1}[\pi, F]$.

If we now consider a deformation A_t of A with multiplictaion f_t, and wish to consider the Hochschild cocomplex $H^\bullet(A_t, A_t)$, we have a coboundary map which we denote δ_t which can now be expressed concicely as for any $F \in C^n(A_t, A_t)$ as

$$\delta_t F = (-1)^{(n-1)}[f_t, F] \tag{4}$$

Do not be frightened by all this theory if you are not used to cohomology! We will only really care about H^n for $n \leq 3$. In fact for a long time only H^n for $n \leq 3$ was actually *used* [8]. The Lie algebra-like structure on $H^\bullet(A, A)$ has been introduced here in order to ease notation later through use of the bracket.

If you are still frightened by Hochschild cohomology, the following example calculations (for H^n with $n \leq 3$) should demystify the contents of the preceding paragraphs.

Example 3. *Consider an algebra A over k.*

- *Consider the identity map $1_A : A \to A$. Clearly $1_A \in C^1(A, A)$ and so we may calculate $\delta 1_A$ via evaluating at any $(a, b) \in A^2$ as*

  $$(\delta 1_A)(a, b) = a1_A(b) - 1_A(ab) + 1_A(a)b = ab - ab + ab = ab = \pi(a, b)$$

  So taking the coboundary of the identity map on A gives us the original multiplication on A.

- *For a any 1-cochain $\phi \in C^1(A, A)$ we have for any $(a, b) \in A^2$ that:*

  $$(\delta\phi)(a, b) = a\phi(b) - \phi(ab) + \phi(a)b.$$

- *For any 2-cochain $F \in C^2(A, A)$ we have for any $(a, b, c) \in A^3$ that:*

  $$(\delta F)(a, b, c) = aF(b, c) - F(ab, c) + F(a, bc) - F(a, b)c.$$

  We can also see this in terms of the bracket operation as

  $$\begin{aligned}(\pi(a, F(b, c)) - \pi(F(a, b), c)) - (F(\pi(a, b), c) - F(a, \pi(b, c))) &= \pi \circ F(a, b, c) - F \circ \pi(a, b, c)\\ &= [\pi, F](a, b, c)\end{aligned}$$

- *For a 3-cochain $\mathcal{G} \in C^3(A, A)$ we have for any $(a, b, c, d) \in A^4$ that:*

  $$(\delta\mathcal{G})(a, b, c, d) = a\mathcal{G}(a, b, c) - \mathcal{G}(ab, c, d) + \mathcal{G}(a, bc, d) - \mathcal{G}(a, b, cd) + \mathcal{G}(a, b, c)d,$$

  and so on.

Given a deformation $f_t = \pi + tF_1 + t^2F_2 + \cdots$ we have not given paid much attention to the terms F_i other than that they are all 2-cochains, so it may well be the case that some F_i are zero. It turns out to be very useful to talk about the *first* nonzero term after π in a deformation. Not so rigorously, the more terms of a deformation which are zero, the more trivial the deformation is. We make this clear in 1.2 after introducing *infinitesimals*.

Definition. *Given a deformation $f_t = \pi + tF_1 + t^2F_2 + \cdots$, the* **infinitesimal** *of f_t is the first nonzero term after π, i.e. the first F_n such that $F_i = 0$ for $0 < i < n$. We call n the* **rank** *of the infinitesimal.*

It turns out that we know a bit more about the structure of infinitesimals than arbitrary terms of f_t as shown in the following proposition.

Proposition 1.1. *Infinitesimals are cocycles.*

Proof. Take a deformation $f_t = \sum_{i\geq 0} t^i F_i$ with infinitesimal F_n (i.e. $F_0 = \pi$ and $F_1 = F_2 = \cdots = F_{n-1} = 0$ and $F_n \not\equiv 0$) for some $n > 0$. Since f_t is associative we have for all $a, b, c \in A$ that $f_t(f_t(a,b),c) = f_t(a, f_t(b,c))$, which when expanded gives us the equation

$$\sum_{m\geq 0}\sum_{i+j=m} t^m F_i(F_j(a,b),c) = \sum_{m\geq 0}\sum_{i+j=m} t^m F_i(a, F_j(b,c)).$$

Upon equating coefficients of t^m for $m \geq 0$ we can find that we have:

$$\sum_{i+j=m} F_i(F_j(a,b),c) = \sum_{i+j=m} F_i(a, F_j(b,c)).$$

Setting $m = n$ we have

$$\sum_{i+j=n} F_i(F_j(a,b),c) = \sum_{i+j=n} F_i(a, F_j(b,c)),$$

but since F_n is the infinitessimal of f_t, the above reduces to the following:

$$F_n(ab,c) + F_n(a,b)c = F_n(a,bc) + F_n(b,c)$$

which when we move all terms to one side we get:

$$aF_n(b,c) - F_n(ab,c) + F_n(a,bc) - F_n(a,b)c = 0 \quad \text{i.e.} \quad (\delta F_n)(a,b,c) = 0,$$

so indeed we have that the infinitesimal is a 3-cocycle. □

Remark. This result holds for deformations of Lie algebras too [2].

Every deformation except the original multiplication has an infinitesimal. Hence an alternate way to define trivial deformation is to say that *a trivial deformation is a deformation equivalent to a deformation with no infinitesimal*. We will now see that some infinitesimals may be removed, thus reducing our deformation to a simpler form.

Proposition 1.2. *A deformation with infinitesimal equal to a coboundary is equivalent to a deformation with infinitesimal of strictly greater rank (and is not equal to a coboundary).*

Proof. Consider a deformation $f_t = \pi + tF_1 + t^2F_2 + \cdots$ of an algebra A with infinitesimal F_n for some $n > 0$. Since we assume F_n is a coboundary, there exists some $\psi_n \in C^1$ such that $\delta\psi_n = F_n$. We define the automorphism

$$\Psi_t := 1 + (-1)^n t^n \psi_n$$

which has inverse

$$\Psi_t^{-1} = 1 - (-1)^n t^n \psi_n + (-1)^n t^{2n} \psi_n^2 - (-1)^n t^{3n} \psi_n^3 + \cdots .$$

Now we express the equivalent deformation $\tilde{f}_t := f_t \circ \Psi_t$ in terms of formal power series as

$$\begin{aligned}(f_t \circ \Psi_t)(a,b) &= \Psi_t^{-1}(f_t(\Psi_t(a), \Psi_t(b))) \\ &= \Psi_t^{-1}\left(f_t(a,b) - (-1)^n t^n (f_t(a, \psi_n(b)) + f_t(\psi_n(a), b)) + t^{2n} f_t(\psi_n(a), \psi_n(b))\right) \\ &= f_t(a,b) + (-1)^n t^n (f_t(a, \psi_n(b)) + f_t(\psi_n(a), b)) - (-1)^n t^n \psi_n(f_t(a,b)) + (\text{degree} > n \text{ terms}).\end{aligned}$$

Now we expand all the f_t in terms of formal power series and then consider the whole expression $f_t \circ \Psi_t$ modulo t^{n+1} to show that the degree n term vanishes. Expanding the above gives:

$$\begin{aligned}(f_t \circ \Psi_t)(a,b) &\equiv ab + t^n F_n(a,b) + (-1)^n t^n a\psi_n(b) + (-1)^n \psi_n(a)b - (-1)^n t^n \psi_n(ab) \pmod{t^{n+1}} \\ &\equiv ab + t^n(F_n(a,b) + (-1)^n(a\psi_n(b) - \psi_n(ab) + \psi_n(a)b)) \pmod{t^{n+1}} \\ &\equiv ab + t^n(F_n(a,b) - \delta\psi_n(a,b)) \pmod{t^{n+1}} \\ &\equiv ab \pmod{t^{n+1}}.\end{aligned}$$

Hence the degree n term of $f_t \circ \Psi_t$ vanishes, giving us a deformation equivalent to f_t but with an infinitesimal with rank strictly greater than n. If the infinitesimal of $f_t \circ \Psi_t$ is a coboundary we may repeat this process increasing the rank of the infinitesimal further. We repeat this process until we arrive at a deformation whose infinitesimal which is not a coboundary. $\square$

The following result is a stronger version of 1.2 and is a first example of how the second Hochschild cohomology group tells us about the deformation theory of an associative algebra.

Corollary 1.3. *If $H^2(A, A) = 0$ then A is rigid.*

Proof. The vanishing of H^2 guarantees infinitesimals of deformations of A to always be coboundaries. (Again, if you are not used to homological algebra this may not be immediately obvious, but this is clear since infinitesimals are *always* cocycles by (1.1), and $H^2 = 0 \iff Z^2 = B^2$) hence, given a deformations f_t, we may repeat the procedure in the proof of (1.2) to dispose of infinitesimals, showing that f_t is in fact equivalent to π, confirming that A is rigid. $\square$

This concludes our introduction to Hochschild cohomology. As mentioned initially, one can spend as much time as one wants on learning about homological algebra and I encourage the interested reader to do so - however for the purposes of this essay we will only need the material presented above. We have already seen that the second Hochschild cohomology group of an associative algebra can tell you a great deal about its deformation theory. We bear this in mind as we now introduce filtered and graded algebras, where we will see that the second Hochschild cohomology group can tell us *even more* about the structure of a filtered associative algebra.

2 Deformations of Filtered and Graded Algebras

In this section we will introduce graded and filtered algebras and study their deformation theory. Using the main result from [3] regarding "jump deformations" of filtered algebras, we will follow Coffee's paper [6] to show that a filtered algebra over a field of characteristic zero is graded when certain conditions on its second Hochschild cohomology group are met.

Filtered and Graded Algebras

Before we define filtered and graded algebras, we note that our favourite (read polynomial) associative algebras $k[x], k[x,y], k[x_1,\ldots,x_n]$ etc. are filtered and graded. The idea behind filtered algebras is that we have a nice grouping of elements in these algebras, or that we can even produce a nice spanning subset of each of these algebras. For example we can filter a polynomial algebra by grouping elements by lower bounds on their total degree. Alternatively we may note that any polynomial can be expressed as a sum of homogeneous polynomials from that same algebra, thus we can write any element of the algebra as a sum of elements from simpler subalgebras. This second idea corresponds to a grading. The processes described above are specific to polynomial algebras, but again, the idea here is to get a hands-on grasp of what we may mean by a filtered algebra or a graded algebra before defining them in the abstract as we shall do next.

Definition. *An algebra A is* **filtered** *if there is a chain of A-submodules*[3] *indexed by natural numbers*[4]*:*

$$A = F^0A \supset F^1A \supset F^2A \supset \cdots$$

such that for any $i, j \geq 0$ we have $F^iAF^jA \subseteq F^{i+j}A$ (i.e. $a \in F^iA$, $b \in F^jB \Rightarrow ab \in F^{i+j}A$).

We refer to the collection $(F^iA)_{i\geq 0}$ as a **filtration**[5] *for A.*

We say that a filtered algebra (or a filtration) is **separated** *if $\bigcap_{i\geq 0} F^iA = 0$.*

Definition. *An algebra B is* **graded** *if B has a filtration $(F^iB)_{i\geq 0}$ and*

$$B \cong \prod_{i\geq 0} B_i$$

where $B_i := F^iB/F^{i+1}B$ for each $i \geq 0$.

Remark. A separated filtered algebra is a topological space. We let $A = F^0A \supset F^1A \supset F^2A \supset \cdots$ be a separated filtered algebra. Letting $\mathcal{B} := \{F^iA \mid i \geq 0\}$ we have that $\mathcal{B}$ is a basis for a topology on A, which we will call the **filtration topology**.

[3] Here we view A as a (left) A-module.

[4] We index using superscripts to distinguish elements F^i of filtrations $(F^i)_i$ and elements $F_i \in C^2(A,A)$.

[5] Strictly speaking, this is a "non-negative filtration" as we index over $0, 1, 2, \ldots$. In general a filtration may be indexed over $\mathbb{Z}$. We will keep calling the former a filtration, throughout this essay

In this definition of graded algebras we see that any graded algebra is filtered, however the converse need not be true in general. We can however create a graded algebra from a filtered one in a manner very similar to graded algebras.

Definition. *Given a filtered algebra A with filtration $(F^iA)_{i\geq 0}$ we define the* **associated graded algebra** *of A to be the completion of the product:*

$$\prod_{i\geq 0} F^iA/F^{i+1}A$$

with respect to the topology induced by the filtration topology. The associated graded algebra is given the symbol $\mathrm{gr}\, A$. In general we do not have $A \cong \mathrm{gr}\, A$.

Note. Taking the completion of a topological algebra in this sense involves some highly non-trivial theory [9] which I will not go through in this essay. There are two main reasons for considering topologically complete associated graded rings. On is that if a filtered ring A is complete in its filtration topology (in the sense of [9]) then A and $\mathrm{gr}\, A$ are *homeomorphic* [4] (although we will not use this fact - but it is a good thing to know). For the purposes of this essay we simply need to know that there exists pathological algebras which cannot be dealt with using the theory we present here.

The main result in Coffee's paper [6] gives a cohomological criterion for when a filtered algebra is isomorphic to its associated graded algebra i.e. when a filtered ring is graded. We will now build up the theory to prove this.

Deformations of Filtered and Graded rings

Before we prove the main result we introduce an incredibly useful result from [3] which involves so-called "jump deformations". Roughly speaking a jump deformation is a deformation A_t of an algebra A in which all the algebras are isomorphic to eachother except perhaps the original one.

Definition. *A* **jump deformation** *is a deformation f_t of an algebra A such that for any two distinct $t, t' \in k \setminus \{0\}$ we have that f_t and $f_{t'}$ are equivalent.*

Equivalently, f_t is a jump deformation if we have an automorphism

$$\Psi_{u,t} = 1 + u\psi_{1,t} + u^2\psi_{2,t} + \cdots$$

such that for any $t' \in k$ there exists some $u \in k$ such that

$$f_{t'} = f_t \circ \Psi_{t,u}.$$

One way to think of jump deformations is to think of them (roughly) as trivial one-parameter deformations of a deformation. That is, if A_t is deformation of A with multiplication f_t, then A_t is a jump deformation if there is a one-parameter family of deformations g_u of A_t such that g_u is equivalent to f_t for all $u \in k$ except perhaps when $t = 0$.

We now have the language to state the main result from [3]:

Theorem 2.1 (Gerstenhaber, 1964)**.** *If A is a filtered algebra, there is a jump deformation A_t of A with $A_1 \cong \operatorname{gr} A$.*

The proof of this theorem follows from a category-theoretic deformation theory of filtered rings developed in [3] and deserves an essay for itself to justify; hence we will take it as fact for now.

Next, we start looking at the Hochschild cohomology of the deformed algebra. That is, if A_t is a deformation of A with multiplication f_t, we now consider $H^n(A_t, A_t)$ as opposed to $H^n(A, A)$. As mentioned after the definition of Hochschild cohomology the Hochschild coboundary operator $\delta_t : C^n(A_t, A_t) \to C^{n+1}(A_t, A_t)$ may be expressed with the bracket notation for any $G \in C^n(A_t, A_t)$ as

$$\delta_t G := (-1)^{n-1}[f_t, G].$$

Note that if $f_t = \pi + tF_1 + t^2F_2 + \cdots$ then we have

$$\delta_t G = (-1)^{n-1} \sum_{i \geq 0} t^i [F_i, G].$$

Given a deformation A_t of A, we have some specific sorts of 2-cocycles in $Z^2(\operatorname{gr} A, \operatorname{gr} A)$, *extendible* ones, and *jumps*. The idea is that there is a correspondence between extendible 2-cocycles of $\operatorname{gr} A$ and 2-cocycles of A_t, and jump cocycles of $\operatorname{gr} A$ correspond to 2-coboundaries of A_t. We make the rigorous definition below:

Definition.

- *A 2-cocycle $z_0 \in Z^2(\operatorname{gr} A, \operatorname{gr} A)$ is said to be an* **extendible cocycle** *if there exists an element $z_t \in Z^2(A_t, A_t)$ of the form $z_t = z_0 + tz_1 + t^2 z_2 + \cdots$ for some $z_1, z_2, \ldots \in Z^2(A, A)$. The element z_t is often called an* **extension***.*
- *We refer to an* **extendible class** *as an cohomology class $[z_0] \in H^2(\operatorname{gr} A, \operatorname{gr} A)$ such that z_0 is extendible.*
- *A 2-cocycle $z_0 \in Z^2(\operatorname{gr} A, \operatorname{gr} A)$ is said to be a* **jump cocycle** *if z_0 is extendible to a coboundary. That is, z_0 is a jump cocycle if there exists some $z_t = z_0 + tz_1 + t^2 z_2 + \cdots \in B^2(A_t, A_t)$ (such that $z_t = \delta_t \eta_t$ for some $\eta_t \in C^1(A_t, A_t)$).*
- *We similarly have the definition of a* **jump class** *$[z_0] \in H^2(\operatorname{gr} A, \operatorname{gr} A)$ being a cohomology class with representative z_0 being a jump cocycle.*

We will denote the sets of extendible classes and jump classes of $\operatorname{gr} A$ as $E^2(\operatorname{gr} A, \operatorname{gr} A)$ and $J^2(\operatorname{gr} A, \operatorname{gr} A)$ respectively.

Remark. We have that $B^2(\operatorname{gr} A, \operatorname{gr} A) \subseteq J^2(\operatorname{gr} A, \operatorname{gr} A) \subseteq E^2(\operatorname{gr} A, \operatorname{gr} A)$. We also note that A_t has underlying algebra $A[[t]]$, which is a $k[[t]]$-algebra. However we will think of A_t as a $k((t))$-algebra so as we can talk about the vector-space *dimension* of A_t. Of course A_t is also a k-algebra, but this is not very helpful when considering its dimension since $\dim_k A_t$ is infinite.

Also remember here that any element in A corresponds to an element in $\operatorname{gr} A$, and so we may be vague about where we take the components z_i from in the above definition.

We now introduce formal derivatives as a method to find extensions of 2-cocycles.

Definition. *Given an n-cochain $c_t = c_0 + tc_1 + t^2c_2 + \cdots \in C^n(A_t, A_t)$. Then the first* **formal derivative** *of c_t*

$$c_t' := c_1 + 2tc_2 + 3t^2c_3 + \cdots = \sum_{i\geq 1} it^{i-1}c_i.$$

Iterating this we have for any $n > 1$, the n-**th formal derivative** *of c_t being:*

$$c_t^{(n)} := \left(c_t^{(n-1)}\right)' \text{ with } c_t^{(1)} := c_t'.$$

We even have an expression for $c_t^{(n)}$ as

$$c_t^{(n)} = \sum_{i\geq n} (i!)t^{i-n}c_i. \tag{5}$$

Note. Using the formal derivative of cochains is what forces us to consider the characteristic of the field being 0. If we were working over a field of some positive characteristic p we would have that the whole formal derivative would disappear after taking at most p derivatives.

Remark. We have some nice formulae for formal derivatives of the following compositions of cochains of an algebra A: Let $\Psi_t = 1_A + t\psi_1 + t^2\psi_2 + \cdots \in C^1(A, A)$, and $f_t = \pi + tF_1 + t^2F_2 + \cdots \in C^2(A, A)$. Then for any $a, b \in A$ we have:

$$(\Psi_t(f_t(a, b)))' = \Psi_t'(f_t(a, b)) + \Psi_t(f_t'(a, b)) \tag{6}$$

and

$$(f_t(\Psi_t(a), \Psi_t(b)))' = f_t'(\Psi_t(a), \Psi_t(b)) + f_t(\Psi_t'(a), \Psi_t(b)) + f_t(\Psi_t(a), \Psi_t'(b)). \tag{7}$$

Proof. For (6) we first the composition on the left hand side as

$$\begin{aligned}\Psi_t(f_t(a, b)) &= \sum_{i\geq 0} t^i\psi_i \left(\sum_{j\geq 0} t^jF_j(a, b)\right) \\ &= \sum_{m\geq 0} \sum_{\substack{i+j=m \\ i,j\geq 0}} t^m\psi_i(F_j(a, b)),\end{aligned}$$

which has formal derivative:

$$(\Psi_t(f_t(a, b)))' = \sum_{m\geq 1} \sum_{\substack{i+j=m \\ i\geq 1 \\ j\geq 0}} mt^{m-1}\psi_i(F_j(a, b)).$$

Similarly the right hand side, expressed as a formal power series, is:

$$\begin{aligned}\Psi_t'(f_t(a,b)) + \Psi_t(f_t'(a,b)) &= \sum_{i\geq 1} it^{i-1}\psi_i\left(\sum_{j\geq 0} t^j F_j(a,b)\right) + \sum_{i\geq 0} t^i\psi_i\left(\sum_{j\geq 1} jt^{j-1}F_j\right) \\ &= \sum_{\substack{i\geq 1\\ j\geq 0}} it^{i+j-1}\psi_i(F_j(a,b)) + \sum_{\substack{i\geq 0\\ j\geq 1}} jt^{i+j-1}\psi_i(F_j(a,b)).\end{aligned}$$

Upon comparing coefficients of these two power-series we see that (6) holds. Showing that (7) holds follows by the same technique hence why the calculations are omitted. □

Proposition 2.2. *Given a deformation $f_t = \pi + t^n F_n + t^{n+1}F_{n+1} + \cdots$ of a filtered algebra A, the infinitesimal, F_n, is a jump cocycle.*

Proof. To show that F_n is a jump cocycle we first need to show that F_n is itself a cocycle of A which extends to a cocycle of A_t with leading term F_n, and then we need to show that the extension of F_n is in fact a coboundary. We already know that F_n is a cocycle by 1.1. We find an appropriate 2-cocycle with F_n as its leading term using derivatives. We have the formal derivative:

$$\begin{aligned}f_t' &= nt^{n-1}F_n + (n+1)t^nF_{n+1} + (n+2)t^{n+1}F_{n+2} + \cdots \\ &= nt^{n-1}\underbrace{\left(F_n + \frac{n+1}{n}tF_{n+1} + \frac{n+2}{n}t^2F_{n+2} + \ \cdots\right)}_{g_t}.\end{aligned}$$

We now show that the term in brackets, g_t, is an extension of F_n (i.e. that g_t is a cocycle of A_t with leading term F_n).

First note that we can write $f_t' = nt^{n-1}g_t$, and that we have the formal derivatives:

$$(f_t(f_t(a,b),c))' = f_t'(f_t(a,b),c) + f_t(f_t'(a,b),c)$$

and

$$(f_t(a,f_t(b,c)))' = f_t'(a,f_t(b,c)) + f_t(a,f_t'(b,c)).$$

By associativity of f_t we have that $f_t(f_t(a,b),c) = f_t(a,f_t(b,c))$, and so taking the formal derivative of both sides of this equation gives:

$$f_t'(f_t(a,b),c) + f_t(f_t'(a,b),c) = f_t'(a,f_t(b,c)) + f_t(a,f_t'(b,c)).$$

Then we rewrite in terms of g_t:

$$nt^{n-1}g_t(f_t(a,b),c) + nt^{n-1}f_t(g_t(a,b),c) = nt^{n-1}g_t(a,f_t(b,c)) + nt^{n-1}f_t(a,g_t(b,c)),$$

Now gathering all terms on one side and factoring out nt^{n-1} gives:

$$nt^{n-1}(g_t(f_t(a,b),c) + f_t(g_t(a,b),c) - g_t(a,f_t(b,c)) - f_t(a,g_t(b,c))) = 0,$$

which in terms of Hochschild coboundary operator δ_t gives us:

$$nt^{n-1}\delta_t g_t(a,b,c) = 0.$$

Since the characteristic of k is zero and $n > 0$, we conclude that $\delta_t g_t = 0$, i.e. that F_n is *extendible*.

To show that F_n is a *jump cocycle* we need to find some $\phi \in C^1(A_t, A_t)$ such that $\delta_t\ \phi = g_t$. This can be done by considering the automorphism (from [3]) $\Phi_t : A_t \to A_1$ which satisfies $f_t(a,b) = \Phi_t^{-1}(f_1(\Phi_t(a), \Phi_t(b)))$ for all $a, b \in A$.

To ease calculations we apply Φ_t to the previous equation, giving us the new equation

$$\Phi_t(f_t(a,b)) = f_1(\Phi_t(a), \Phi_t(b))$$

Taking the formal derivative of both sides of this equation with respect to t using formulae (6) and (7) we have:

$$\Phi'_t(f_t(a,b)) + \Phi_t(f'_t(a,b)) = f_1(\Phi'_t(a), \Phi_t(b)) + f_1(\Phi_t(a), \Phi'_t(b)). \tag{8}$$

Noting that since the 2-cochain, f_1, on the right hand side does not depend on t so its formal derivative is zero.

Applying Φ_t^{-1} to both sides of (8) and solving for $f'_t(a,b)$ gives us

$$\begin{aligned} f'_t(a,b) &= \Phi_t^{-1}(f_1(\Phi'_t(a), \Phi_t(b))) - \Phi_t^{-1}\Phi'_t(f_t(a,b)) + \Phi_t^{-1}(f_1(\Phi_t(a), \Phi'_t(b))) \\ &= \Phi_t^{-1}(f_1(\Phi_t\Phi_t^{-1}\Phi'_t(a), \Phi_t(b))) - \Phi_t^{-1}\Phi'_t(f_t(a,b)) + \Phi_t^{-1}(f_1(\Phi_t(a), \Phi_t\Phi_t^{-1}\Phi'_t(b))). \end{aligned}$$

Since Φ_t is an automorphism, the first and last terms on the right hand side of the above equation simplify to $f_t(\Phi_t^{-1}\Phi'_t(a), b)$ and $f_t(a, \Phi_t^{-1}\Phi'_t(b))$ respectively, giving us the following expression for $f'_t(a,b)$:

$$f'_t(a,b) = f_t(\Phi_t^{-1}\Phi'_t(a), b) - \Phi_t^{-1}\Phi'_t(f_t(a,b)) + f_t(a, \Phi_t^{-1}\Phi'_t(b)) = (\delta_t(\Phi_t^{-1}\Phi'_t))(a,b).$$

Hence f'_t is a coboundary of A_t, showing that g_t is one too (since δ_t is $k((t))$-linear), and so F_n is indeed a jump cocycle. □

We now present the main results in [6].

Lemma 2.3 (Coffee 1972). *If $\dim_k H^2(\operatorname{gr} A, \operatorname{gr} A)$ is finite, then*

$$\dim_{k((t))} H^2(A_t, A_t) = \dim_k \frac{E^2(\operatorname{gr} A, \operatorname{gr} A)}{J^2(\operatorname{gr} A, \operatorname{gr} A)}.$$

Proof. Let $m := \dim_k H^2(\operatorname{gr} A, \operatorname{gr} A)$. Choose a $k((t))$-basis for $H^2(A_t, A_t)$ of the form $[z_t^i]$, with $z_t^i = z_0 + tz_1^i + t^2 z_2^i + \cdots$ such that $\{[z_0^i] \mid i = 1, \ldots, m\}$ is k-linearly independent. Since each $z_0^i \in Z^2(\operatorname{gr} A, \operatorname{gr} A)$ is extendible we have that

$$\dim_{k((t))} H^2(A_t, A_t) \leq \dim_k \frac{E^2(\operatorname{gr} A, \operatorname{gr} A)}{J^2(\operatorname{gr} A, \operatorname{gr} A)}.$$

For the other inequality we consider the map $E^2(\operatorname{gr} A, \operatorname{gr} A) \to H^2(A_t, A_t)$ defined by $[z_0] \mapsto [z_t]$ (clearly for each extendible $z_0 \in Z^2(\operatorname{gr} A, \operatorname{gr} A)$ there by definitions exists $z_t = z_0 + tz_1 + t^2 z_2 + \cdots$ in $Z^2(A_t, A_t)$, so we can think of this map as the map which takes extendible cocycles to their extensions). You may check that this map is k-linear. Let θ be the linear extension of the preceding map to $k((t))$. The kernel of θ is exactly $J^2(\mathrm{gr}\, A, \mathrm{gr}\, A)$ since:

$$\ker\theta = \{[z_0] \ \in E^2(\mathrm{gr}\, A, \mathrm{gr}\, A) \mid [z_t] \ = 0 \ \in H^2(A_t, A_t)\},$$

but $[z_t] = 0 \in H^2(A_t, A_t)$ just says $z_t \in B^2(A_t, A_t)$. Hence there exists some $\phi \in C^1(A_t, A_t)$ such that $\delta_t \phi = z_t$, showing that z_0 is a jump cocycle. Hence $\ker\theta \subseteq J^2(\mathrm{gr}\, A, \mathrm{gr}\, A)$. It is clear that any jump class maps to zero in $H^2(A_t, A_t)$ since the extension of a jump cocycle is by definition a coboundary. Hence by the first isomorphism theorem we have

$$\frac{E^2(\mathrm{gr}\, A, \mathrm{gr}\, A)}{J^2(\mathrm{gr}\, A, \mathrm{gr}\, A)} \cong \mathrm{im}\,\theta \subset H^2(A_t, A_t)$$

as $k((t))$-modules. Note that the dimension of $\frac{E^2(\mathrm{gr}\, A, \mathrm{gr}\, A)}{J^2(\mathrm{gr}\, A, \mathrm{gr}\, A)}$ as a k-vector space equals its dimension as a $k((t))$-vector space (as the cochains in the Hochschild cohomology of $\mathrm{gr}\ A$ do not involve t), thus establishing the other inequality:

$$\dim_k \frac{E^2(\mathrm{gr}\, A, \mathrm{gr}\, A)}{J^2(\mathrm{gr}\, A, \mathrm{gr}\, A)} \leq \dim_{k((t))} H^2(A_t, A_t).$$

□

Now we can finally put together all of the machinery we have developed to prove a fundamental result by Coffee:

Theorem 2.4 (Coffee 1972)**.** *Let A be a separeted complete filtered algebra over a field k of characteristic 0. If $\dim_k H^2(A, A) = \dim_k H^2(\mathrm{gr}\, A, \mathrm{gr}\, A)$ and is finite, then A is isomorphic to $\mathrm{gr}\, A$.*

Proof. First we reduce the problem via 2.1 which implies that it is enough to show that if $\dim_k H^2(\mathrm{gr}\, A, \mathrm{gr}\, A) = \dim_{k((t))} H^2(A_t, A_t)$ is finite and $t \neq 0$ then $A_t \cong \mathrm{gr}\, A[[t]]$ with multiplication f_0.

By 2.3 we know that $\dim_{k((t))} H^2(A_t, A_t) = \dim_k \frac{E^2(\mathrm{gr}\, A, \mathrm{gr}\, A)}{J^2(\mathrm{gr}\, A, \mathrm{gr}\, A)}$. In general we know that $\dim_k \frac{E^2(\mathrm{gr}\, A, \mathrm{gr}\, A)}{J^2(\mathrm{gr}\, A, \mathrm{gr}\, A)} \leq \dim_k H^2(\mathrm{gr}\, A, \mathrm{gr}\, A)$, since $E^2(\mathrm{gr}\, A, \mathrm{gr}\, A) \subseteq H^2(\mathrm{gr}\, A, \mathrm{gr}\, A)$. Hence this new assumption that $\dim_k H^2(\mathrm{gr}\, A, \mathrm{gr}\, A) = \dim_{k((t))} H^2(A_t, A_t)$ gives us:

$$\dim_k H^2(\mathrm{gr}\, A, \mathrm{gr}\, A) = \dim_k \frac{E^2(\mathrm{gr}\, A, \mathrm{gr}\, A)}{J^2(\mathrm{gr}\, A, \mathrm{gr}\, A)},$$

which forces us to have $J^2(\mathrm{gr}\, A, \mathrm{gr}\, A) = B^2(\mathrm{gr}\, A, \mathrm{gr}\, A)$. (This is clear since if $J^2(\mathrm{gr}\, A, \mathrm{gr}\, A) \supsetneq B^2(\mathrm{gr}\, A, \mathrm{gr}\, A)$, then we would have a strict inequality $\dim_k \frac{E^2(\mathrm{gr}\, A, \mathrm{gr}\, A)}{J^2(\mathrm{gr}\, A, \mathrm{gr}\, A)} < \dim_k H^2(\mathrm{gr}\, A, \mathrm{gr}\, A)$ contradicting 2.3).

We denote the multiplication in the deformation A_t by f_t with infinitesimal F_n for some $n \geq 1$. We have that F_n is a coboundary, since f_t is a jump deformation (and so represents a jump class). Hence there exists some $\phi_n \in C^1(\operatorname{gr} A, \operatorname{gr} A)$ such that $F_n = \delta\phi_n$. Hence the automorphism $\Phi_t = 1_{A_t} - t^n\phi_n$ gives us an automorphism $\Phi_t : A_t \to \operatorname{gr} A[[t]]$ where $\operatorname{gr} A[[t]]$ has multiplication with infinitesimal of strictly greater rank. We may repeat this procedure as we did in 1.2 to produce an automorphism $A_t \to \operatorname{gr} A[[t]]$ where $\operatorname{gr} A[[t]]$ has multiplication f_0 as required. □

3 Conclusion

Here ends our very quick tour of deformations of algebras. We have seen plenty of examples of deformations, and a connection to geometry. We have seen Hochschild cohomology and showed that calculating the second Hochschild cohomology group of an associative algebra tells us a great deal about its deformation theory. We have also seen how we can use deformations of algebras as a tool to develop strictly non-deformation-theoretic criteria for showing that a filtered algebra is graded when working over a field of characteristic zero. Working with deformations of algebras over fields of positive characteristic is, as expected, a more cumbersome procedure and can be explored in most of Gerstenhaber's papers on deformation theory. The flavour of this theory is substantially different to this essay and to my own liking and so I have avoided including fields of positive characteristic in this essay, however I encourage the interested reader to start with [2] and branch further at their leisure.

I hope this has been an enjoyable pedagogical short survey of deformations of algebras, and that the reader leaves this essay with ,at the very least, a firm grasp of how much structure there is to play with in something as nice as a polynomial algebra.

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